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A method for estimating errors in calculated strains

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Abstract—Accurate strain estimation requires tests to identify whether natural distributions of strain markers satisfy the assumptions of the method used. To evaluate the capabilities of different strain calculation and statistical methods for this purpose, simulated ellipse distributions of varying axial ratios and orientations are strained and analyzed. If bedding and cleavage cannot be measured, statistical tests on strained ellipse distributions do not produce useful results because of the inseparable effects of strain, initial ellipticity and initial preferred orientations. Tests on data destrained using averaged ellipse matrix values can indicate initial preferred orientations and the nature of the axial ratio distribution. The most useful tests are the chi-square tests of the ellipticity and orientation distributions of the destrained data. The Rayleigh test cannot identify initial preferred orientations in this type of destrained data.

INTRODUCTION

MANY methods exist for calculating strain from deformed elliptical markers (see review by Babaie 1986). Most methods assume the strain markers were initially elliptical, with uniformly varying ellipticities and orientations. However, a preferred orientation is common in undeformed rocks. It is important to identify strained samples where the randomness assumption fails, so that the strain estimates may be viewed with suspicion. Because the effects of strain and initial preferred orientation can be similar, this identification is not always possible or quantitatively precise.

Existing methods for detecting initial preferred orientations require some combination of data on strain history and the orientations of bedding, extension and/or cleavage, and initial markers (Borradaile 1984, 1987, Yu & Zheng 1984, Wheeler 1986b). The method described here is less sensitive but more general. It can be used where bedding and cleavage are not visible, or where the elliptical strain markers do not record the complete bedding deformation (e.g. the pre-lithification deformation of Hudleston 1976). It applies to generally oriented sections in three dimensions, where the cleavage trace does not necessarily parallel the extension direction of the clasts (Ramsay 1967, Borradaile 1987).

The statistics used here to test the strain data are based on the inferred initial shapes and orientations of particles. In the destrained state, the complexly interacting effects of strain magnitude and orientation are avoided ('destrained' refers to strained ellipses from which the measured strain has been removed; Borradaile 1984). Useful parameters are the deviation of the ellipse orientations from a random distribution, their average axial ratio, the dispersion of the axial ratios and the type of distribution fit by those ratios. Strain analyses yielding anomalous values (e.g. statistically non-random initial orientations) can be rejected on the grounds that fundamental assumptions of the strain determination technique are being violated. The destrained clast shapes and orientations can be compared with actual measurements on undeformed rocks (e.g. Boulter 1976, Holst 1982, Wheeler 1984). In this report, specified initial distributions of ellipses (Fig. 1a) are numerically strained and then analyzed to illustrate the behaviour of the statistical parameters (Fig. 1 and Table 1).

STATISTICAL TESTS FOR STRAINED SAMPLES

Methods of strain determination

An objective, programmable, rotation invariant method is needed to efficiently calculate the strain ellipse from a sample with deformed elliptical particles. Paterson (1983) compared different methods of strain determination on cut faces. Of the algebraic methods, that of Shimamoto & Ikeda (1976) gave the best strain estimate. It was the most precise of all methods, particularly with small sample sizes. Only the theta-curve method of Lisle (1977b) generally gave lower strain estimates, which should be the most accurate (Siddans 1980). However, it is less precise, and it suffers from its subjective dependency on the chosen theta-curve spacing as well as the fineness of strain increments (Wheeler 1984). Robin's (1977) method, favored by De Paor (1988) for numerical determinations, produces similar results (Babaie 1986). For these reasons, strains are calculated with the Shimamoto & Ikeda (1976) method below (see also Wheeler 1984). Criticisms of their technique (Wheeler 1986a, De Paor 1988) only apply to the three-dimensional case.

Destraining

Useful statistical tests of strain estimates must be based on the destrained state, circumventing the strained state where the effects of strain, initial ellipticii-

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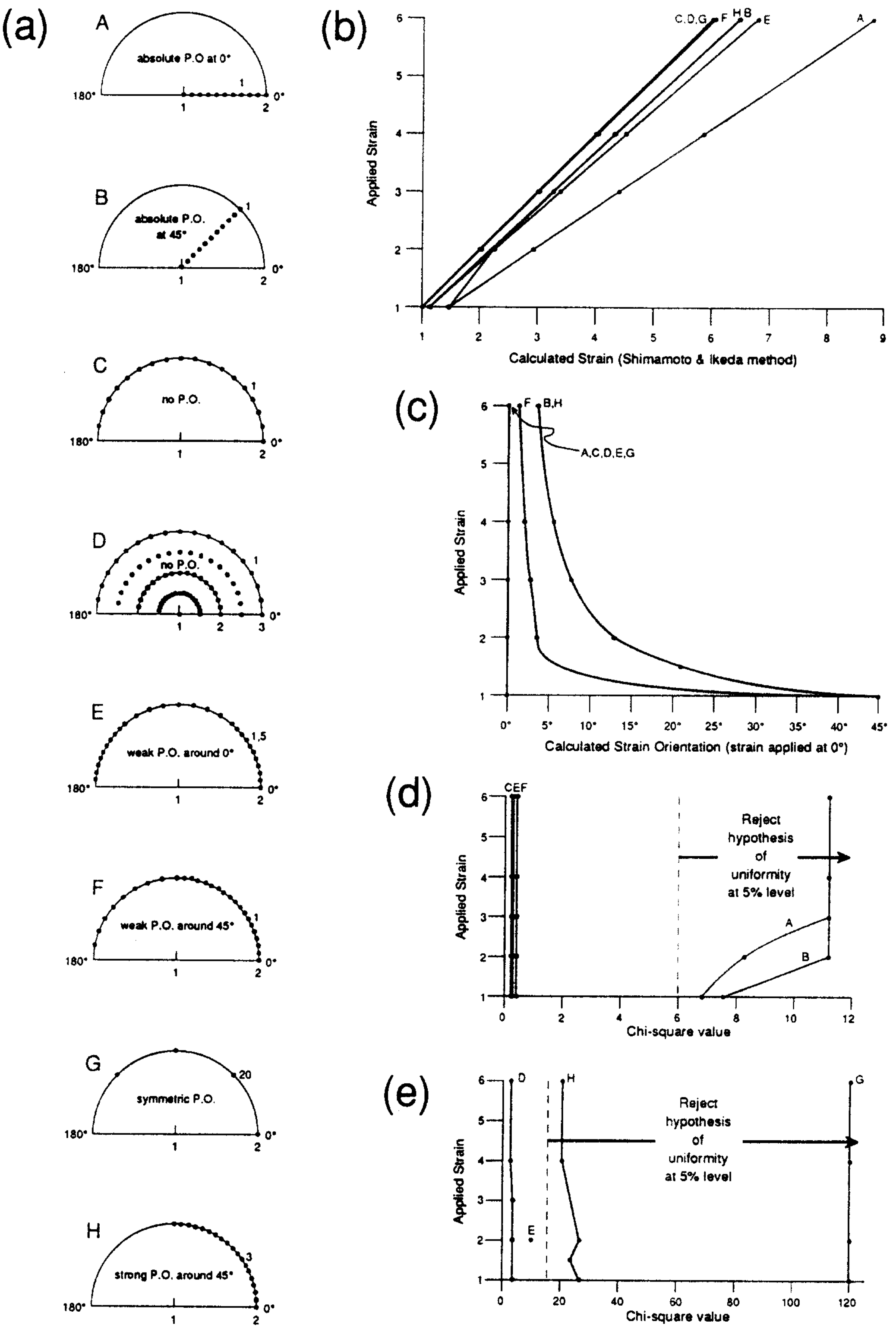

Fig. 1. (a) Initial ellipse distributions used for the strain simulations, from data in the first four columns of Table 1. Axial ratios plot radially, major-axis orientations plot tangentially. The number on the middle right indicates the number of ellipses represented by each dot. 'P.O.' means preferred orientation. (b) Plot of applied strains against those calculated by the Shimamoto & Ikeda (1976) method for each distribution in (a). (c) Major-axis orientations of the calculated strain ellipses. (d) & (e) Results of the chi-square test for uniformity of destrained ellipse orientations for distributions with <50 and ≥50 ellipses, respectively

ties and preferred orientations are difficult to disentangle. Methods of destraining which rely on minimizing a chi-square variable to calculate the strain (Lisle 1977a, Peach & Lisle 1979) are inaccurate because they consider only the individual orientations of the ellipses (ϕ), but not their corresponding ellipticities (R_f). In mathematical simulations the vector mean defining the strain orientation may vary more than 10° from the true strain ellipse axis (Borradaile 1984).

As a preferable alternative to chi-square minimizations, methods employing the mean ellipse matrix use all of the information available from the strain markers.

Table 1. Summary of results from simulations of strained ellipse distributions

Test	No. of Ellipses	Axial Ratios	Initial Orientations	Applied Strain	Harmonic Mean	R_s Elliott	R_s S&I	ϕ_s S&I	R_d	L_{mean}	Chi-square values θ	Chi-square values R_i	Chi-square values $\ln(R_i)$	StdDev R_i
A	11	1-2, 0.1	all 0°	1.0	1.43	1.47	1.47	0.00°	1.24	0.09	6.82 R	0.27	0.27	0.14
	•	•		2.0	2.86	2.93	2.93	0.00°	1.24	0.09	8.27 R	0.27	0.27	0.14
	•	•		3.0	4.30	4.40	4.40	0.00°	1.24	0.09	11.18 R	0.27	0.27	0.14
	•	•		4.0	5.72	5.86	5.86	0.00°	1.24	0.09	11.18 R	0.27	0.27	0.14
	•	•		6.0	8.59	8.79	8.79	0.00°	1.24	0.09	11.18 R	0.27	0.27	0.14
B	11	1-2, 0.1	all 45°	1.0	1.43	1.47	1.47	45.00°	1.24	0.09	7.55 R	0.27	0.27	0.14
	•			2.0	2.28	2.26	2.24	12.95°	1.24	0.09	11.18 R	0.27	0.27	0.14
	•			3.0	3.34	3.32	3.27	7.64°	1.24	0.09	11.18 R	0.27	0.27	0.14
	•			4.0	4.42	4.40	4.33	5.50°	1.24	0.09	11.18 R	0.27	0.27	0.14
	•			6.0	6.59	6.59	6.47	3.56°	1.24	0.09	11.18 R	0.27	0.27	0.14
C	18	all 2.0	0-170°, 10°	1.0	2.00	1.00	1.00	0.00°	2.00	0.00	0.22	0.22	0.22	0.00
	•			2.0	2.24	2.10	2.00	0.00°	2.00	0.00	0.22	2.44	2.44	0.00
	•			3.0	3.11	3.23	3.00	0.00°	2.00	0.00	0.22	7.78 R	7.78 R	0.00
	•			4.0	4.08	4.36	4.00	0.00°	2.00	0.00	0.22	2.00	2.00	0.00
	•			6.0	6.05	6.63	6.00	0.00°	2.00	0.00	0.22	7.33 R	4.67 R	0.00
D	90	1-3, 0.5	0-170°, 10°	1.0	1.72	1.00	1.00	0.00°	2.10	0.00	3.56	72.0 R	90.0 R	0.71
	•			2.0	2.35	2.11	2.00	0.00°	2.10	0.00	3.56	72.9 R	90.0 R	0.71
	•			3.0	3.15	3.25	3.00	0.00°	2.10	0.00	3.56	74.0 R	90.0 R	0.71
	•			4.0	4.10	4.40	4.00	0.00°	2.10	0.00	2.72	72.0 R	90.0 R	0.71
	•			6.0	6.06	6.70	6.00	0.00°	2.10	0.00	2.72	82.9 R	90.0 R	0.71
E	28	all 2.0	0-170°,10°;	1.0	2.00	1.15	1.13	0.00°	1.98	0.09	0.29	0.86	0.86	0.17
	•		−45-45°,10°	2.0	2.55	2.42	2.26	0.00°	1.98	0.09	0.29	0.86	0.86	0.17
	140			2.0	2.55	2.42	2.26	0.00°	1.98	0.09	10.00	99.29 R	99.29 R	0.16
	28			3.0	3.60	3.72	3.39	0.00°	1.98	0.09	0.29	0.86	0.86	0.17
	•			4.0	4.72	5.02	4.52	0.00°	1.98	0.09	0.29	0.86	0.86	0.17
	•			6.0	7.01	7.61	6.78	0.00°	1.98	0.09	0.29	0.86	0.86	0.17
F	27	all 2.0	0-170°,10°;	1.0	2.00	1.16	1.14	45.00°	1.97	0.10	0.41	2.19	2.19	0.18
	•		5-85°,10°	2.0	2.24	2.13	2.03	4.84°	1.97	0.10	0.41	2.19	2.19	0.18
	•			3.0	3.11	3.25	3.03	2.74°	1.97	0.10	0.41	2.19	2.19	0.18
	•			4.0	4.08	4.38	4.04	1.95°	1.97	0.10	0.41	2.19	2.19	0.18
	•			6.0	6.05	6.64	6.05	1.26°	1.97	0.10	0.41	2.19	2.19	0.18
G	80	all 2.0	0°, 45°,	1.0	2.00	1.00	1.00	0.00°	2.00	0.00	120.0 R	220.0 R	220.0 R	0.00
	•		90°, 135°	2.0	2.03	2.10	2.00	0.00°	2.00	0.00	120.0 R	220.0 R	220.0 R	0.00
	•			4.0	3.94	4.35	4.00	0.00°	2.00	0.00	120.0 R	220.0 R	220.0 R	0.00
	•			6.0	5.88	6.60	6.00	0.00°	2.00	0.00	120.0 R	220.0 R	220.0 R	0.00
H	57	all 2.0	0-90°, 5°	1.0	2.00	1.52	1.46	45.00°	1.76	0.17	26.68 R	26.68 R	20.37 R	0.29
	•			1.5	2.05	1.81	1.75	20.45°	1.76	0.17	23.53 R	26.68 R	20.37 R	0.29
	•			2.0	2.20	2.31	2.23	12.85°	1.76	0.17	26.68 R	26.68 R	20.37 R	0.29
	•			4.0	4.02	4.48	4.32	5.45°	1.76	0.17	20.37 R	26.68 R	20.37 R	0.29
	•			6.0	5.97	6.71	6.46	3.53°	1.76	0.17	20.37 R	26.68 R	20.37 R	0.29

In the 'Axial Ratios' and 'Initial Orientations' columns, the first numbers give the range of values, and the numbers following the comma give the value increment within that range.

Symbols and abbreviations are: R_s, calculated strain value; Elliott and S&I, calculated by Elliott (1970) and Shimamoto & Ikeda (1976) methods, respectively; ϕ_s, calculated maximum extension orientation; R_d, average initial ellipticity; L_{mean}, mean resultant length of orientation vectors; R_i, initial axial ratio; Std Dev, standard deviation. R after numbers in the chi-square columns signifies rejection of the null hypothesis at the 5% level.

Using the calculated strain (by the Shimamoto & Ikeda method) for the section, equations from Elliott (1970) and Holst (1982) are used here to destrain the measured ellipses, yielding their inferred initial orientations, θ, and axial ratios, R_i. Several statistical tests may then be used to determine the nature of the destrained ellipse distribution. Tests based on principal axis orientations are the most widely employed, so they will be discussed first.

Axial orientations

A fundamental assumption in strain calculations is that the original ellipses were oriented randomly. For an unknown mean direction, the Rayleigh test is the best choice to test the null hypothesis of uniformity of orientations (randomness for a large population) against an unspecified probability density distribution (Curray 1956, Durand & Greenwood 1958). The test depends on L_{mean}. First consider the *destrained* orientations as unit vectors from a 180° range, then double them. L_{mean} is the vector magnitude of the mean of the vector components. A zero L_{mean} indicates the orientations are balanced around the circle. Uniformity is rejected for large values of L_{mean} (critical values given by Mardia 1972, appendix 2.5).

The Rayleigh test cannot distinguish uniformity (e.g. C and D of Fig. 1a) from angularly symmetrical or balanced concentrations (G of Fig. 1a). In this case, the non-uniformity may be revealed by a chi-square test using the frequency with which θ observations fall in separate angular sectors. My data are divided into four equal angular classes for less than 50 ellipses, and 10 classes for more, so that at least five orientations can be expected in each class (Peach & Lisle 1979). The number of ellipses in each sector is compared with the expected number for a uniform distribution, and the squared, normalized difference is summed. Uniformity is rejected for large values when compared with standard chi-square tables.

Borradaile (1984) suggested a runs test to sense clustering of orientations in adjacent classes. However, Borradaile found the runs test often rejected the randomness hypothesis for populations of less than 75, even though his strain simulation *started* with random distributions. Such a test does not seem useful for common sample sizes.

Axial ratios

A full description of the initial state of particles includes the distribution of their axial ratios (Boulter 1976). Their average initial ellipticity, R_d, is found as follows. Wheeler (1984) devised a single measure of the initial ellipticity of deformed markers, termed the "distribution spread invariant", J. J is the square root of the determinant of the averaged final ellipse matrix of Shimamoto & Ikeda (1976), with the desirable characteristic of strain invariance. J only measures the *average* initial ellipticity, with no direct information on its variability (contrary to Wheeler 1984, 1986a). J is related to R_d by

$$J = \cosh (\log_e R_d)$$

(Wheeler 1984, equation A25), and so

$$R_d = J + (J^2 - 1)^{1/2}.$$

As with J, R_d does not measure the dispersion of the initial axial ratios. A large R_d *likely* indicates a large dispersion, because most distributions include near spherical particles. However, this implied measure of variance is clearly not in a standard form comparable to standard statistical parameters.

Measures of the variance are more informative when compared with a probability distribution. Obviously the R_i cannot be uniformly distributed, since they have a minimum by definition at $R_i = 1$, but no specific maximum. Two other distributions seem likely. The normal distribution commonly describes the variation of random variables. However, with a distinct minimum axial ratio and a stronger concentration usual near that minimum (e.g. Pfiffner 1980, Borradaile 1987), a log-normal distribution better describes that skewness. For example, a normal distribution was accepted for only 34% of 90 sample faces from an Archean greenstone belt, but the log-normal distribution was accepted for 77% of them (Schultz-Ela & Hudleston, manuscript in review).

A chi-square test similar to that described above compares the observed R_i distribution with normal and log-normal distributions. To relate the data to standard tables of the distributions, the R_i and $\ln (R_i)$ are converted to standard normal form. Normal curve segments of equal probability bound classes. The null hypothesis of a normal or log-normal distribution is rejected for large values of the chi-square variables. Because both the mean and standard deviation are estimated from the data, the degrees of freedom are three less than the number of classes (e.g. Davis 1986).

STRAIN SIMULATION RESULTS

To investigate the accuracy of the strain estimates for ellipses with various initial ellipticities and orientations, I started with a known ellipse distribution, applied a known strain, and analyzed the results. With no loss of generality, the maximum extension direction of the applied strain is always at 0°. Figure 1 and Table 1 summarize results for some of the simulated strains. Note that tests C and D have uniform initial orientations, and the remaining tests have preferred orientations.

Strain determination

Table 1 emphasizes the behavior of the various strain measures. The Shimamoto & Ikeda (1976) method yields the most accurate calculated strain values (R_s), even for distributions with initial preferred orientations. For a strong preferred orientation, the estimates for applied strains greater than 2 are less than 18% in error (tests B, E, F and H), except when a single orientation parallels the strain axis (test A). The corrected mean from the plot of Wheeler (1984) is calculated from similar equations, so would produce identical results. The higher values of the Elliott plot mean (Table 1, Elliott column) reflect the uncorrected distortion inherent in that method (Holst 1982, Wheeler 1984). Strains are overestimated for all tests with preferred orientations except G, comprising clusters of ellipses at 45° intervals. Here the harmonic mean actually *underestimates* the true strain, which in this case equals the matrix average. Possible natural examples with such an underestimate are reported by Babaie (1986). To underscore the importance of testing the destrained data, note the almost identical strains calculated for tests B and H, despite their radically different initial ellipse distributions.

The estimate of the maximum extension direction, ϕ_s, varies substantially (Fig. 1c). For preferred orientations (e.g. tests B, F and H) the error decreases with increasing strain. The true average initial ellipticity is underestimated by R_d (Table 1), but not as severely as by arithmetic, geometric and harmonic means.

Variations of the chi-square values with strain (Table 1) largely stem from round-off errors for θ values that coincide with class boundaries at 45° intervals. Thus, particularly for small samples the chi-square values are not rotationally invariant, the grouping effects noted by Harvey & Ferguson (1981). The number of ellipses also strongly affects the chi-square values (e.g. test E). A preferred orientation does not change the values much, because destraining removes some of the non-uniformity.

Orientation tests

For fewer than 50 ellipses, the orientation chi-square value (Table 1, θ column) has two degrees of freedom, while the R_i measures have only one. For 50 or more ellipses, the degrees of freedom are eight and seven, respectively. Using 5% critical values, the chi-square test rejects the null hypothesis of a uniform distribution in tests A, B, G and H (Figs. 1d & e, and Table 1), which are the tests with the strongest initial preferred orientations.

The Rayleigh test (L_{mean} in Table 1) does not reject the hypothesis of uniformity for any of the simulations. Indeed, a destraining procedure based on the assumption of uniform initial orientations will minimize L_{mean} (Harvey & Ferguson 1981). As the destraining removes more than the true strain for a preferred orientation, ellipses with a low initial axial ratio elongate nearly perpendicular to their initial orientation (as shown by R_d less than the true average initial ellipticity for tests A, B, E, F and H). In the L_{mean} calculation these 'overstrained' orientations partially cancel out those still in the original preferred orientation, and the test accepts uniformity. In this regard, the chi-square test is more sensitive to initial preferred orientations, although natural examples occur in which uniformity is rejected by the Rayleigh but not the chi-square test (Schultz-Ela & Hudleston, manuscript in review). For the same reasons, calculations based on chi-square tests of orientation (e.g. Peach & Lisle 1979, Borradaile 1984, 1987) would overestimate the strain more than the present method by attempting to return an initial preferred orientation distribution to non-existent uniformity.

Axial ratio tests

The initial axial ratios were tested for fit to a normal or log-normal distribution (Table 1), although they were not chosen to simulate such distributions. The variable results of test C are a numerical artifact of a uniform R_i and a vanishingly small standard deviation. The results overall illustrate the strong sensitivity of the tests to sample size. Samples with more than about 50 ellipses yield reliable chi-square results for R_i, even with initial preferred orientations.

The standard deviation of R_i for distributions with an initial preferred orientation is too small, reflecting the excessive destraining which concentrates R_i around unity as some of the ellipse elongations shift to a new direction.

CONCLUSIONS

Straining known ellipse distributions of varying axial ratios and orientations illustrates the behaviour of different strain calculation and statistical methods. The averaged ellipse matrix method of Shimamoto & Ikeda (1976) calculates the known strains more accurately than methods based only on axial ratios or orientations. Statistical tests on strained ellipse distributions do not produce useful results because of the inseparable effects of strain, initial ellipticity and initial preferred orientations. Tests on data destrained with the Shimamoto & Ikeda (1976) values can indicate initial preferred orientations and the nature of the axial ratio distribution, and identify strain markers which do not meet the requirements for accurate strain estimation. The most useful tests are the chi-square tests of the orientation and ellipticity distributions. The Rayleigh test cannot identify initial preferred orientations in this type of destrained data.

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